

# ON THE DISTRIBUTION OF SOME EULER-MAHONIAN STATISTICS

ALEXANDER BURSTEIN

**ABSTRACT.** We give a direct combinatorial proof of the equidistribution of two pairs of permutation statistics,  $(\text{des}, \text{aid})$  and  $(\text{lec}, \text{inv})$ , which have been previously shown to have the same joint distribution as  $(\text{exc}, \text{maj})$ , the major index and the number of excedances of a permutation. Moreover, the triple  $(\text{pix}, \text{lec}, \text{inv})$  was shown to have the same distribution as  $(\text{fix}, \text{exc}, \text{maj})$ , where  $\text{fix}$  is the number of fixed points of a permutation. We define a new statistic  $\text{aix}$  so that our bijection maps  $(\text{pix}, \text{lec}, \text{inv})$  to  $(\text{aix}, \text{des}, \text{aid})$ . We also find an Eulerian partner  $\text{das}$  for a Mahonian statistic  $\text{mix}$  defined using mesh patterns, so that  $(\text{das}, \text{mix})$  is equidistributed with  $(\text{des}, \text{inv})$ .

## 1. INTRODUCTION

A *combinatorial statistic* on a set  $S$  is a map  $\mathbf{f} : S \rightarrow \mathbb{N}^m$  for some integer  $m \geq 0$ . The *distribution* of  $\mathbf{f}$  is the map  $\mathbf{d}_{\mathbf{f}} : \mathbb{N}^m \rightarrow \mathbb{N}$  with  $\mathbf{d}_{\mathbf{f}}(\mathbf{i}) = |\mathbf{f}^{-1}(\mathbf{i})|$  for  $\mathbf{i} \in \mathbb{N}^m$ , where  $|\mathbf{f}^{-1}(\mathbf{i})|$  is the number of objects  $s \in S$  such that  $\mathbf{f}(s) = \mathbf{i}$ . We say that statistics  $\mathbf{f}$  and  $\mathbf{g}$  are *equidistributed* and write  $\mathbf{f} \sim \mathbf{g}$  if  $\mathbf{d}_{\mathbf{f}} = \mathbf{d}_{\mathbf{g}}$ .

Let  $\mathfrak{S}_n$  be the set of permutations of  $[n] = \{1, \dots, n\}$ . The four classic combinatorial statistics on  $\mathfrak{S}_n$ , the number of *descents*,  $\text{des}$ , the number of *excedances*,  $\text{exc}$ , the number of *inversions*,  $\text{inv}$ , and the *major index*,  $\text{maj}$ , are defined as follows:

$$\begin{aligned} \text{Des } \pi &= \{i : \pi(i) > \pi(i+1)\}, & \text{des } \pi &= |\text{Des } \pi|, \\ \text{Exc } \pi &= \{i : \pi(i) > i\}, & \text{exc } \pi &= |\text{Exc } \pi|, \\ \text{Inv } \pi &= \{(i, j) : i < j \text{ and } \pi(i) > \pi(j)\}, & \text{inv } \pi &= |\text{Inv } \pi|, \\ & & \text{maj } \pi &= \sum_{i \in \text{Des } \pi} i. \end{aligned}$$

The sets  $\text{Des } \pi$  is called the *descent set* of  $\pi$ , and its elements are called *descents*. If  $i$  is a descent of  $\pi$ , then  $\pi(i)$  and  $\pi(i+1)$  are called *descent top* and *descent bottom*, respectively. The terminology for the other two sets,  $\text{Inv } \pi$  and  $\text{Exc } \pi$ , is similar. When the context is unambiguous, we may refer to the pair  $\pi(i)\pi(i+1)$  as a descent or the pair  $(\pi(i), \pi(j))$  as an inversion.

A statistic with the same distribution as  $\text{des}$  (such as  $\text{exc}$ ) is called *Eulerian*, and a statistic with the same distribution as  $\text{inv}$  (such as  $\text{maj}$  [9]) is called *Mahonian*. If  $\text{eul}$  is Eulerian and  $\text{mah}$  is Mahonian, then the pair  $(\text{eul}, \text{mah})$  is called an Euler-Mahonian statistic.

A problem frequently considered since [2] is as follows: given a known Euler-Mahonian statistic  $(\text{eul}_1, \text{mah}_1)$  and another Eulerian (resp. Mahonian) statistic  $\text{eul}_2$  (resp.  $\text{mah}_2$ ), to find its Mahonian (resp. Eulerian) partner  $\text{mah}_2$  (resp.  $\text{eul}_2$ ) so that  $(\text{eul}_1, \text{mah}_1) \sim (\text{eul}_2, \text{mah}_2)$ . In this paper, we will give two bijective proofs of equidistribution of two such pairs of bivariate statistics. In Section 2,

---

*Date:* February 14, 2014.

*Key words and phrases.* Permutation statistic, Eulerian, Mahonian, major index, excedance, descent, pattern.

we give a direct proof of a bijection between two statistics previously shown to have the same distribution as  $(\text{exc}, \text{maj})$ , and in Section 3 we find an Eulerian partner  $\text{das}$  for a statistic  $\text{mix}$  recently defined by Brändén and Claesson [1] using mesh patterns so that  $(\text{das}, \text{mix}) \sim (\text{des}, \text{inv})$ .

$$2. (\text{des}, \text{aid}) \sim (\text{lec}, \text{inv})$$

Of the four pairs  $(\text{eul}_1, \text{mah}_1)$  involving  $\text{des}$  or  $\text{exc}$  and  $\text{inv}$  or  $\text{maj}$ , the last to be considered was the pair  $(\text{exc}, \text{maj})$ . First, Shareshian and Wachs [10] found a Mahonian statistic  $\text{aid}$  such that  $(\text{exc}, \text{maj}) \sim (\text{des}, \text{aid})$ , and soon afterwards Foata and Han [3] proved that  $(\text{exc}, \text{maj}) \sim (\text{lec}, \text{inv})$  for an Eulerian statistic  $\text{lec}$  defined earlier by Gessel [4] and related to the hook factorization of a permutation. In fact, Foata and Han proved a more refined result that  $(\text{fix}, \text{exc}, \text{maj}) \sim (\text{pix}, \text{lec}, \text{inv})$ , where  $\text{fix } \pi$  is the number of fixed points of  $\pi$  and  $\text{pix } \pi$  is another statistic related to hook factorization of  $\pi$ .

We will now define the  $\text{aid}$ ,  $\text{lec}$  and  $\text{pix}$ .

**Definition 2.1.** An inversion  $(i, j) \in \text{Inv } \pi$  is *admissible* if either  $\pi(j) < \pi(j+1)$  or  $\pi(j) > \pi(k)$  for some  $i < k < j$ . Let  $\text{Ai } \pi$  be the set of admissible inversions of  $\pi$ , and let

$$\text{ai } \pi = |\text{Ai } \pi|, \quad \text{aid } \pi = \text{ai } \pi + \text{des } \pi.$$

**Definition 2.2.** A string  $w = w_1 w_2 \dots w_r$  ( $r \geq 2$ ), over a totally ordered alphabet is a *hook* if  $w_1 > w_2 \leq w_3 \leq \dots \leq w_r$ . Every string  $\pi$  over  $\mathbb{N}$  (and hence any permutation  $\pi$ ) can be decomposed uniquely [4] as  $\pi = \pi_0 \pi_1 \dots \pi_k$  ( $k \geq 0$ ), where  $\pi_0$  is an nondecreasing string and each of  $\pi_i$ ,  $1 \leq i \leq k$ , is a hook. Then  $\pi_0 \pi_1 \dots \pi_k$  is called the *hook factorization* of  $\pi$ .

It is easy to see that the hook factorization is unique for any  $\pi$ , since either  $\pi = \pi_0$  or we can recursively find the rightmost hook of  $\pi$ , which starts with the rightmost descent top of  $\pi$ . The statistics  $\text{lec}$  and  $\text{pix}$  are defined as follows:

$$\text{lec } \pi = \sum_{i=1}^k \text{inv } \pi_i, \quad \text{pix } \pi = |\pi_0|,$$

where  $|\pi_0|$  is the length of  $\pi_0$ .

Shareshian and Wachs [10] gave a proof of  $(\text{des}, \text{aid}) \sim (\text{exc}, \text{maj})$  using tools from poset topology such as lexicographic shellability. Subsequently, Foata and Han [3] gave a two-step proof of  $(\text{fix}, \text{exc}, \text{maj}) \sim (\text{pix}, \text{lec}, \text{inv})$ . The first step was a bijection on  $\mathfrak{S}_n$  showing that  $(\text{fix}, \text{exc}, \text{maj}) \sim (\text{pix}, \text{lec}, \text{imaj})$  (and, in fact, a more refined result that  $(\text{fix}, \text{exc}, \text{des}, \text{maj}) \sim (\text{pix}, \text{lec}, \text{ides}, \text{imaj})$ ), where  $\text{imaj}(\pi) = \text{maj}(\pi^{-1})$  and  $\text{ides}(\pi) = \text{des}(\pi^{-1})$ , using Lyndon words and the word analogs of Kim-Zeng [6] permutation decomposition and hook factorization. The second step was a bijection on  $\mathfrak{S}_n$  showing that  $(\text{pix}, \text{lec}, \text{imaj}) \sim (\text{pix}, \text{lec}, \text{inv})$ .

Somewhat surprisingly, a direct bijective proof of  $(\text{des}, \text{aid}) \sim (\text{lec}, \text{inv})$  is simpler than any of the bijections mentioned above. We give such a proof and, in fact, find a new statistic  $\text{aix}$  that is a  $\text{fix}$ -partner for  $(\text{des}, \text{aid})$ , i.e. such that  $(\text{aix}, \text{des}, \text{aid}) \sim (\text{pix}, \text{lec}, \text{inv}) \sim (\text{fix}, \text{exc}, \text{maj})$ .

The statistic  $\text{aix}$  is defined as follows. Consider the set  $\mathbb{N}^*$  of all strings in  $\mathbb{N}$ . Given a string  $\pi \in \mathbb{N}^*$ , let  $m$  be the smallest letter in  $\pi$  and let  $\alpha$  be the maximal left prefix of  $\pi$  not containing  $m$ , so that  $\pi = \alpha m \beta$  for some string  $\beta$ . Then we recursively define  $\text{aix } \emptyset = 0$  and, for  $\pi \neq \emptyset$ ,

$$\begin{aligned} (2.1a) \quad & \text{aix } \alpha, & \text{if } \alpha \neq \emptyset, \beta \neq \emptyset, \\ (2.1b) \quad & 1 + \text{aix } \beta, & \text{if } \alpha = \emptyset, \\ (2.1c) \quad & 0, & \text{if } \beta = \emptyset. \end{aligned}$$

For example,  $\text{aix}(2589637\underline{14}) = \text{aix}(2589637) = 1 + \text{aix}(5896\underline{37}) = 1 + \text{aix}(5896) = 1 + 1 + \text{aix}(89\underline{6}) = 1 + 1 + 0 = 2$  (the smallest letters at each step are underlined).

**Proposition 2.3.** *For any  $\pi \in \mathbb{N}^*$ , we have  $\text{aix } \pi \leq 1 + \text{pix } \pi$ .*

*Proof.* The value of  $\text{aix } \pi$  is at most the length of  $\rho$ , the maximal nondecreasing left prefix of  $\pi$ . Since the leftmost hook of  $\pi$  starts either at the leftmost descent or at the second leftmost descent (only if it immediately follows the leftmost descent), it follows that the length of  $\rho$  is either  $\text{pix } \pi$  or  $1 + \text{pix } \pi$ .  $\square$

We also note that computations of statistics  $\text{inv}, \text{lec}, \text{pix}, \text{aid}, \text{des}, \text{aix}$ , involves only comparisons of values of letters or values of positions, but not values of a letter and a position (as in computation of  $\text{exc}$ ), so that these statistics can be extended to any string of distinct letters.

**2.1. The bijection.** Let  $S$  be a set of distinct letters and  $k \notin S$  be such that  $S \cup \{k\}$  is totally ordered. Let  $\tau$  be a permutation of  $S$ . Let  $m$  be the smallest letter in  $S \cup \{k\}$ . Define a permutation  $f(k, \tau)$  of  $S \cup \{k\}$  recursively as follows:  $f(k, \emptyset) = k$  and

$$\begin{aligned} (2.2a) \quad & f(k, \alpha)m\beta, & \text{if } \tau = \alpha m \beta, k > m, \alpha \neq \emptyset, \beta \neq \emptyset, \\ (2.2b) \quad & f(k, \beta)m, & \text{if } \tau = m\beta, k > m, \\ (2.2c) \quad & km\alpha, & \text{if } \tau = \alpha m, k > m, \\ (2.2d) \quad & k\tau, & \text{if } k = m. \end{aligned}$$

Now, for  $\pi \in \mathfrak{S}_n$ , define  $\phi_0(\pi) = \emptyset$  and  $\phi_k(\pi) = f(\pi(n-k+1), \phi_{k-1}(\pi))$ ,  $k = 1, \dots, n$ . Finally, let  $\phi(\pi) = \phi_n(\pi) \in \mathfrak{S}_n$ . It is straightforward to see that  $f$ , and thus,  $\phi$ , are bijections.

Let  $\text{ini } \pi = \pi(1)$ . Then we have that

**Theorem 2.4.**  $(\text{ini}, \text{aix}, \text{des}, \text{aid}) \phi(\pi) = (\text{ini}, \text{pix}, \text{lec}, \text{inv}) \pi$ .

We will split the proof of the theorem into several parts.

**Lemma 2.5.**  $\text{ini } \phi(\pi) = \text{ini } \pi$ .

*Proof.* Note that  $f(k, \emptyset) = k$ , so by the definition of  $f$  and induction on the size of  $\tau$  we get that  $f(k, \tau)$  starts with  $k$ . Thus,  $\phi(\pi)$  starts with  $\pi(1)$ .  $\square$

Given a string  $\pi$  over a totally ordered alphabet define  $k$ -suffix of  $\pi$ ,  $s_k(\pi)$ , to be the block of  $k$  rightmost letters of  $\pi$ . Also, define  $\pi_{<k}$  (resp.  $\pi_{>k}$ ) to be the subsequence of  $\pi$  consisting of letters of  $\pi$  that are less (resp. greater) than  $k$ .

**Lemma 2.6.**  $\text{aid } f(k, \tau) = \text{aid } \tau + |\tau_{<k}|$ .

*Proof.* We will prove this lemma by induction on the length of  $\tau$ . Clearly, the lemma is true for  $\tau = \emptyset$ . Assume that the lemma holds for all strings of distinct letters of length less than  $|\tau|$ . Let  $m = \min \tau$  and consider each case in the definition of  $f(k, \tau)$ .

*Case (a).* Suppose that  $\tau = \alpha m \beta$ ,  $k > m$ ,  $\alpha \neq \emptyset$ ,  $\beta \neq \emptyset$ . Then  $f(k, \tau) = f(k, \alpha)m\beta$ , so by Lemma 2.5,  $f(k, \alpha m \beta) = k \hat{\alpha} m \beta$  for some permutation  $\hat{\alpha}$  of  $\alpha$ . By induction (since  $|\alpha| < |\tau|$ ), we have

$$\text{aid } f(k, \alpha) = \text{aid } \alpha + |\alpha_{<k}|.$$

Consider the inversions  $ab$  in  $\tau$  that are from  $\alpha$  to  $m\beta$ , i.e. those where the inversion top is  $a \in \alpha$  and the inversion bottom is  $b \in m\beta$  (so  $a > b$ ). If  $b = m$ , then it is followed by an ascent, and hence any inversion with inversion bottom  $m$  is admissible (and the number of such (admissible)

inversions in  $\tau$  is  $|\alpha|$ ). If  $b \in \beta$ , then  $m < b$  and  $m$  is between  $a$  and  $b$  in  $\tau$ , so the inversion  $ab$  is admissible. Thus, all inversions from  $\alpha$  to  $m\beta$  are admissible.

Since  $\hat{\alpha}$  is a permutation of  $\alpha$ , we likewise have that all inversions in  $f(k, \tau)$  from  $\hat{\alpha}$  to  $m\beta$  are admissible, and in fact, are the same inversions as the inversions from  $\alpha$  to  $m\beta$  in  $\tau$ . Moreover, since  $\alpha > m$  (i.e. every letter in  $\alpha$  is greater than  $m$ ) and  $f$  does not change the suffix  $m\beta$  of  $\tau$ , it follows that the number of admissible inversions in  $m\beta$  and the number of descents with descent bottoms in  $m\beta$  are the same in  $\tau$  and  $f(k, \tau)$ .

Thus, the only remaining pairs left to consider are inversions from  $k$  to  $m\beta$ . As above, we see that all inversions from  $k$  to  $m\beta$  are admissible, and the number of such inversions is exactly  $|m\beta_{<k}|$ . Therefore,

$$\text{aid } f(k, \tau) - \text{aid } \tau = |\alpha_{<k}| + |m\beta_{<k}| = |\alpha_{<k}m\beta_{<k}| = |\tau_{<k}|,$$

as desired.

*Case (b).* Suppose that  $\tau = m\beta$  and  $k > m$ . Then  $f(k, \tau) = f(k, \beta)m = k\hat{\beta}m$  for some permutation  $\hat{\beta}$  of  $\beta$ . As before, we have by induction that

$$\text{aid } f(k, \beta) = \text{aid } \beta + |\beta_{<k}|.$$

Since  $k\hat{\beta} > m$  and  $m$  is last in  $k\hat{\beta}m$ , it follows that no admissible inversion ends on  $m$ . Thus,  $\text{ai } f(k, \beta)m = \text{ai } f(k, \beta)$  and  $\text{des } f(k, \beta)m = \text{des } f(k, \beta) + 1$ , where 1 counts the last descent to  $m$ . Finally,  $\text{aid } \tau = \text{aid } m\beta = \text{aid } \beta$  since  $m < \beta$  and hence no inversion (or descent) of  $\tau$  begins with  $m$ . Therefore,

$$\text{aid } f(k, \tau) = \text{aid } f(k, \beta) + 1 = \text{aid } \beta + |\beta_{<k}| + 1 = \text{aid } m\beta + |m\beta_{<k}| = \text{aid } \tau + |\tau_{<k}|.$$

*Case (c).* Suppose that  $\tau = \alpha m$ ,  $k > m$ . Then  $f(k, \tau) = km\alpha$ . Thus, the descents of  $f(k, \tau)$  are obtained from descents of  $\tau$  by replacing the descent from the right letter of  $\alpha$  to  $m$  with the descent  $km$ , so  $\text{des } f(k, \tau) = \text{des } \tau$ . As Case (b), no admissible inversion of  $\tau$  ends on  $m$ , and, as in Cases (a) and (b), all inversions from  $k$  to  $m\alpha$  are admissible. Thus,

$$\text{ai } f(k, \tau) = \text{ai } km\alpha = \text{ai } m\alpha + |m\alpha_{<k}| = \text{ai } \alpha m + |\alpha_{<k}m| = \text{ai } \tau + |\tau_{<k}|,$$

so

$$\text{aid } f(k, \tau) = \text{ai } f(k, \tau) + \text{des } f(k, \tau) = \text{ai } \tau + |\tau_{<k}| + \text{des } \tau = \text{aid } \tau + |\tau_{<k}|.$$

*Case (d).* If  $k < \tau$ , then no inversion (or descent) of  $f(k, \tau) = k\tau$  starts with  $k$ , and  $|\tau_{<k}| = 0$ , so  $\text{aid } f(k, \tau) = \text{aid } \tau = \text{aid } \tau + \tau_{<k}$ . This ends the proof.  $\square$

**Lemma 2.7.**  $\text{aid } \phi(\pi) = \text{inv } \pi$ .

*Proof.* Applying Lemma 2.6 repeatedly, we obtain

$$\text{aid } \phi(\pi) = \sum_{k=0}^{n-1} |\phi_k(\pi)_{<\pi(n-k)}|.$$

But each  $\phi_k(\pi)$  is a permutation of  $s_k(\pi)$ , so

$$\text{aid } \phi(\pi) = \sum_{k=0}^{n-1} |s_k(\pi)_{<\pi(n-k)}|.$$

Each summand on the right is the number of inversions of  $\pi$  with inversion top  $\pi(n-k)$ . Summing over  $k = 0, 1, \dots, n-1$ , we get  $\text{aid } \phi(\pi) = \text{inv}(\pi)$ , as desired.  $\square$

Consider the descents of  $\tau$  and  $f(k, \tau)$  in each case of the definition of  $f$ . In case (2.2a), we have  $\alpha > m$  and  $f(k, \tau) = f(k, \alpha m \beta) = f(k, \alpha) m \beta$ , so the descent bottoms in the right prefix  $m \beta$  of both  $\tau$  and  $f(k, \tau)$  are the same, and hence

$$\text{des } f(k, \tau) - \text{des } \tau = \text{des } f(k, \alpha) - \text{des } \alpha.$$

Note that in this case  $\text{aix } \tau = \text{aix } \alpha$  and  $\text{aix } f(k, \tau) = \text{aix } f(k, \alpha)$ .

In case (2.2b),  $\text{des } \tau = \text{des } m \beta = \text{des } \beta$  since  $m < \beta$ . However,  $\text{des } f(k, \tau) = \text{des } f(k, \beta) m = \text{des } f(k, \beta) + 1$  since  $f(k, \beta) = k \hat{\beta}$  for some permutation  $\hat{\beta}$  of  $\beta$ . Thus,

$$\text{des } f(k, \tau) - \text{des } \tau = \text{des } f(k, \beta) - \text{des } \beta + 1.$$

Note that in this case  $\text{aix } f(k, \tau) = 0$ , and  $\text{aix } \tau = 1 + \text{aix } \beta > 0$ .

In case (2.2c), let  $a$  be the last letter of  $\alpha$ . Then the descents of  $f(k, \tau) = km\alpha$ ,  $\alpha \neq \emptyset$  are obtained from the descents of  $\tau = \alpha m$  by replacing the descent  $am$  with the descent  $km$ . Thus,  $\text{des } f(k, \tau) = \text{des } \tau = \text{des } \alpha + 1$ , and hence

$$\text{des } f(k, \tau) - \text{des } \tau = 0.$$

Note that in this case  $\text{aix } \tau = 0$  and  $\text{aix } f(k, \tau) = \text{aix } k = 1 = 1 + \text{aix } \tau$ .

In case (2.2d),  $f(k, \tau) = k\tau$ , and  $k < \tau$ , so  $\text{des } f(k, \tau) = \text{des } \tau$ , and hence again

$$\text{des } f(k, \tau) - \text{des } \tau = 0.$$

Note that in this case  $\text{aix } f(k, \tau) = \text{aix } \tau + 1 > 0$ .

Finally,  $\text{des } f(k, \emptyset) - \text{des } \emptyset = 1 - 0 = 1$ . Thus, we can see by induction on the length of  $\tau$  that

$$\text{des } f(k, \tau) - \text{des } \tau \geq 0$$

for any string  $\tau$  of distinct letters, and the difference stays the same or increases by 1 with each application of rules (2.2a) or (2.2b), respectively.

**Lemma 2.8.** *We have  $\text{des } f(k, \tau) = \text{des } \tau$  if and only if  $\text{aix } f(k, \tau) = \text{aix } \tau + 1 > 0$ , and  $\text{des } f(k, \tau) > \text{des } \tau$  if and only if  $\text{aix } f(k, \tau) = 0$ .*

*Proof. Case 1.* Suppose that  $\text{des } f(k, \tau) = \text{des } \tau$ . Then it follows from the above argument that the computation of  $f(k, \tau)$  involves no application of (2.2b), i.e. a repeated application of (2.2a) (possibly zero times) followed by a single application of (2.2c) or (2.2d) or  $f(k, \emptyset) = k$ . The conditions in the case (2.2a) are the same as in the case (2.1a), so applying (2.2a) repeatedly, we obtain either

- a prefix  $\alpha' m'$  of  $\tau$  such that  $\alpha \neq \emptyset$ ,  $\alpha' > m'$ ,  $k > m'$ ,  $\text{aix } \tau = \text{aix } \alpha' m'$  and  $\text{aix } f(k, \tau) = \text{aix } f(k, \alpha' m')$ , or
- a prefix  $\alpha''$  of  $\tau$  such that  $k < \alpha''$ ,  $\text{aix } \tau = \text{aix } \alpha''$  and  $\text{aix } f(k, \tau) = \text{aix } f(k, \alpha'')$ .

In the former case, we have  $\text{aix } \tau = \text{aix } \alpha' m' = 0$  and  $\text{aix } f(k, \tau) = \text{aix } f(k, \alpha' m') = \text{aix } k m' \alpha' = \text{aix } k = 1 = 1 + \text{aix } \tau$ . In the latter case, we have  $\text{aix } f(k, \alpha'') = \text{aix } k \alpha'' = 1 + \text{aix } \alpha'' = 1 + \text{aix } \tau$ . Thus, in either case,  $\text{des } f(k, \tau) = \text{des } \tau$  implies  $\text{aix } f(k, \tau) = \text{aix } \tau + 1$ . The converse is proved similarly.

*Case 2.* Suppose that  $\text{des } f(k, \tau) > \text{des } \tau$ . Then the computation of  $f(k, \tau)$  starts with a repeated application of (2.2a) (possibly zero times) followed by an application of (2.2b) (after which the process may still continue). Thus, as before, after repeated application of (2.2a), we obtain a prefix  $m' \beta'$  of  $\tau$  such that  $k > m'$ ,  $m' < \beta'$  and  $\text{aix } f(k, \tau) = \text{aix } f(k, m' \beta') = \text{aix } f(k, \beta') m'$ .

But  $f(k, \beta') = k\hat{\beta}'$  for some permutation  $\hat{\beta}'$  of  $\beta'$ , so  $f(k, \beta') > m'$ , and hence  $\text{aix } f(k, \beta')m' = 0$ , which in turn implies that  $\text{aix } f(k, \tau) = 0$ , as desired. The converse is proved similarly.  $\square$

**Lemma 2.9.** *If  $\text{aix } \tau = 0$ , then for any  $k$ , we have  $\text{aix } f(k, \tau) = 1$  and  $\text{des } f(k, \tau) = \text{des } \tau$ .*

*Proof.* The lemma is obviously true for  $\tau = \emptyset$ . Suppose  $\tau \neq \emptyset$ . Since  $\text{aix } \tau = 0$ , it follows that  $\tau = \alpha m_0 m_1 \beta_1 \dots m_r \beta_r$ , where  $\alpha \neq \emptyset$ ,  $\alpha > m_0$ ,  $\beta_i \neq \emptyset$  and  $m_i < \beta_i$  for all  $i = 1, \dots, r$ , and  $m_0 > m_1 > \dots > m_r$ . If  $k > m_0$ , then applying (2.2a) repeatedly followed by (2.2c), we obtain

$$f(k, \tau) = f(k, \alpha m_0 m_1 \beta_1 \dots m_r \beta_r) = f(k, \alpha m_0) m_1 \beta_1 \dots m_r \beta_r = k m_0 \alpha m_1 \beta_1 \dots m_r \beta_r$$

so that  $\text{aix } f(k, \tau) = \text{aix } k m_0 \alpha = \text{aix } k = 1$ . Also, all descent bottoms of  $\tau$  and  $f(k, \tau)$  are the same (including  $m_0$ ), so  $\text{des } f(k, \tau) = \text{des } \tau$ .

Suppose that  $k < m_0$ , and let  $j$  be the maximal such that  $k < m_j$ . Then  $k < \alpha m_0 m_1 \beta_1 \dots m_j \beta_j$ , so

$$\begin{aligned} f(k, \tau) &= f(k, \alpha m_0 m_1 \beta_1 \dots m_r \beta_r) \\ &= f(k, \alpha m_0 m_1 \beta_1 \dots m_j \beta_j) m_{j+1} \beta_{j+1} \dots m_r \beta_r \\ &= k \alpha m_0 m_1 \beta_1 \dots m_j \beta_j m_{j+1} \beta_{j+1} \dots m_r \beta_r \\ &= k \tau. \end{aligned}$$

Therefore,  $f(k, \tau) = k\tau$  starts with an ascent, so  $\text{des } f(k, \tau) = \text{des } k\tau = \text{des } \tau$  and hence  $\text{aix } f(k, \tau) = 1 + \text{aix } \tau = 1$  by Lemma 2.8.  $\square$

**Lemma 2.10.** *Suppose that  $\text{aix } f(k, \tau) = 0$  and  $\tau = f(l, \sigma)$  for some letter  $l$  and string  $\sigma$ . Then  $\text{des } f(k, \tau) = 1 + \text{des } f(k, \sigma)$ .*

*Proof.* Note that  $\text{aix } \tau \geq 1$  since otherwise  $\text{aix } f(k, \tau) = 1$ . In particular,  $\tau \neq \emptyset$ , so indeed there is a letter  $l$  and a string  $\sigma$  such that  $\tau = f(l, \sigma)$ .

Since  $\tau = f(l, \sigma)$ , it follows that  $\tau$  starts with  $l$ . Let  $l = m_0 > m_1 > \dots > m_r$  be the left-to-right minima of  $\tau$ . Then  $\tau = m_0 \tau_0 m_1 \tau_1 \dots m_r \tau_r$  with  $\tau_i > m_i$  for all  $i = 0, 1, \dots, r$ . We also have that  $\tau_i \neq \emptyset$  for  $i \geq 1$  since otherwise  $\text{aix } \tau = 0$ . Therefore,

$$f(k, \tau) = f(k, m_0 \tau_0 m_1 \tau_1 \dots m_r \tau_r) = f(k, l \tau_0) m_1 \tau_1 \dots m_r \tau_r,$$

so  $\text{aix } f(k, \tau) = \text{aix } f(k, l \tau_0)$ . If  $k < l$ , then  $k < l \tau_0$ , so  $f(k, l \tau_0) = k l \tau_0$  and

$$\text{aix } f(k, l \tau_0) = 1 + \text{aix } l \tau_0 = 1 + \text{aix } \tau > 0,$$

which contradicts our assumption. Therefore,  $k > l$ .

Since  $\text{aix } f(l, \sigma) = \text{aix } \tau > 0$ , it follows that the recursive computation of  $f(l, \sigma)$  involves no application of (2.2b). Thus, we have two cases:

- $\sigma = \alpha l_1 \beta_1 \dots l_s \beta_s$ , where  $l > l_1 > \dots > l_s$ ,  $\alpha \neq \emptyset$ ,  $\alpha > l$ ,  $\beta_i \neq \emptyset$  and  $\beta_i > l_i$  for  $i = 1, \dots, s$ .
- $\sigma = \alpha l_0 l_1 \beta_1 \dots l_s \beta_s$ , where  $l > l_0 > l_1 > \dots > l_s$ ,  $\alpha \neq \emptyset$ ,  $\alpha > l$ ,  $\beta_i \neq \emptyset$  and  $\beta_i > l_i$  for  $i = 1, \dots, s$ .

Let  $\beta = l_1 \beta_1 \dots l_s \beta_s$ . In the first case, we have

$$\begin{aligned} \tau &= f(l, \sigma) = f(l, \alpha \beta) = f(l, \alpha) \beta = l \alpha \beta = l \sigma \\ f(k, \tau) &= f(k, l \alpha \beta) = f(k, l \alpha) \beta = f(k, \alpha) l \beta \\ f(k, \sigma) &= f(k, \alpha \beta) = f(k, \alpha) \beta. \end{aligned}$$

Note that  $\text{ini } \beta = l_1 < l$ . Also note that  $f(k, \alpha)l\beta = k\hat{\alpha}l\beta$  for some permutation  $\hat{\alpha}$  of  $\alpha$ . Since  $\alpha > l$ , it follows that  $\hat{\alpha} > l$ . Let  $a$  be the last letter of  $f(k, \alpha)$ . Then the descents of  $f(k, \alpha)l\beta$  are obtained from the descents of  $f(k, \alpha)\beta$  by replacing the descent  $al_1$  with the descents  $al$  and  $ll_1$ . Therefore, we have  $\text{des } f(k, \tau) = \text{des } f(k, \sigma) + 1$  as desired.

In the second case, we have

$$\begin{aligned}\tau &= f(l, \sigma) = f(l, \alpha l_0 \beta) = f(l, \alpha l_0) \beta = ll_0 \alpha \beta \\ f(k, \tau) &= f(k, ll_0 \alpha \beta) = f(k, ll_0 \alpha) \beta = f(k, l) l_0 \alpha \beta = k ll_0 \alpha \beta \\ f(k, \sigma) &= f(k, \alpha l_0 \beta) = f(k, \alpha l_0) \beta = k l_0 \alpha \beta.\end{aligned}$$

Since  $k > l > l_0$ , it is easy to see that  $\text{des } f(k, \tau) = \text{des } f(k, \sigma) + 1$ . This ends the proof.  $\square$

**Lemma 2.11.**  $(\text{aix}, \text{des}) \phi(\pi) = (\text{pix}, \text{lec}) \pi$ .

*Proof.* The proof is by induction on the length of  $\pi$ . The result is obviously true for  $\pi = \emptyset$ . Define  $g(k, \tau) = k\tau$  for a string  $\tau$  of distinct elements and an element  $k$  not in the alphabet of  $\tau$ . Then it is easy to see that the results of Lemmas 2.8, 2.9 and 2.10 hold if we replace  $f$  with  $g$ ,  $\text{aix}$  with  $\text{pix}$ , and  $\text{des}$  with  $\text{lec}$ . This implies the lemma and thus finishes the proof of Theorem 2.4.  $\square$

*Remark 2.12.* We note that a statistic  $\text{rix}$  similar to  $\text{aix}$  (up to an easy transformation) has been independently defined by Z. Lin [8].

It would be interesting to construct a direct bijection on permutations that maps  $(\text{aix}, \text{des}, \text{aid})$  to  $(\text{fix}, \text{exc}, \text{maj})$ .

*Remark 2.13.* *Rawlings major index*  $\text{rmaj}$  is a Mahonian statistics that interpolates between  $\text{maj}$  and  $\text{inv}$ , and is defined as follows:

$$\begin{aligned}\text{Des}_r(\pi) &= \{i \in \text{Des}(\pi) : \pi(i) - \pi(i+1) \geq r\} \\ \text{Inv}_r(\pi) &= \{(i, j) \in \text{Inv}(\pi) : \pi(i) - \pi(j) < r\} \\ \text{rmaj}(\pi) &= \sum_{i \in \text{Des}_r(\pi)} i + |\text{Inv}_r(\pi)|\end{aligned}$$

Note that on  $\mathfrak{S}_n$ ,  $\text{lmaj} = \text{maj}$ ,  $\text{nmaj} = \text{inv}$ , and  $|\text{Inv}_2(\pi)| = \text{ides}(\pi) = \text{des}(\pi^{-1})$ . It is known [11] that  $(\text{ides}, 2\text{maj}) \sim (\text{exc}, \text{maj})$ . It would be interesting to find a  $\text{fix}$ -partner  $2\text{fix}$  for  $(\text{ides}, 2\text{maj})$  so that  $(2\text{fix}, \text{ides}, 2\text{maj}) \sim (\text{fix}, \text{exc}, \text{maj})$ . Continuing in the same vein, for  $3 \leq r \leq n-1$ , it would be interesting to find the interpolating statistics  $r\text{fix}$  and  $r\text{exc}$  so that  $(\text{fix}, \text{exc}, \text{maj}) \sim (r\text{fix}, r\text{exc}, \text{rmaj}) \sim (\text{pix}, \text{lec}, \text{inv})$ .

### 3. $(\text{das}, \text{mix}) \sim (\text{des}, \text{inv})$

A Mahonian statistic  $\text{mix}$  counting some inversions and some noninversions has been defined by P. Brändén, A. Claesson [1]. Even though it was originally defined using *mesh patterns*, it may be easily defined without using those. The statistic  $\text{mix}$  counts pairs defined on a permutation  $\pi$  as follows:

- inversions  $(\pi(i), \pi(j))$  such that  $\pi(i)$  is a left-to-right maximum of  $\pi$ , and
- non-inversions  $(\pi(i), \pi(j))$  such that there is a (left-to-right-maximum)  $\pi(k)$  with  $k < i$  and  $\pi(k) > \pi(j)$ .

We note that, in fact, our definition of  $\text{mix}$  is the reversal of the  $\text{mix}$  as originally defined in [1]. However, we think our definition is preferable, since we have  $\text{mix}(id_n) = 0$ , rather than  $\text{mix}(id_n) = n - 1$  under the original definition.

There is also a direct bijection given in [1] that takes  $\text{inv}$  to  $\text{mix}$ . Making the necessary minor changes to account for the difference in definitions mentioned above, we describe it as follows.

Let  $M, I \in [n]$  be such that  $|M| = |I|$  and  $n \in M$ ,  $1 \in I$ , and let  $\mathfrak{S}_n(M, I)$  be the set of permutations in  $\mathfrak{S}_n$  that have left-to-right maxima exactly at the positions indexed by  $I$ , and set of values of the left-to-right maxima equal to  $M$ .

Let  $M = \{m_1 < \dots < m_k\}$ , and let  $B_i$  be the set of entries of  $\pi$  that are smaller than and to the right of  $m_i$ . Also, for  $S \subseteq [n]$ , let  $\psi_S(\pi)$  be the result of reversing the subword of  $\pi$  that is a permutation on  $S$ . Then define

$$\psi = \psi_{B_1} \circ \psi_{B_2 \cap B_1} \circ \dots \circ \psi_{B_{k-1}} \circ \psi_{B_k \cap B_{k-1}} \circ \psi_{B_k}.$$

Then we have [1] that  $\psi$  is an involution and  $\text{mix } \psi(\pi) = \text{inv } \pi$  (and vice versa).

We observe that there is a natural Eulerian partner  $\text{das}$  (a mix of descents and ascents) for  $\text{mix}$  such that is a  $(\text{das}, \text{mix}) \sim (\text{des}, \text{inv})$ . Let  $\text{das } \pi$  be the number of positions  $i \in [n - 1]$  of  $\pi$  such that

- $\pi(i)\pi(i + 1)$  is a descent, and  $\pi(i)$  is a left-to-right maximum of  $\pi$ , or
- $\pi(i)\pi(i + 1)$  is an ascent, and there is a (left-to-right-maximum)  $\pi(k)$  with  $k < i$  and  $\pi(k) > \pi(i + 1)$ .

**Theorem 3.1.**  $(\text{das}, \text{mix}) \psi(\pi) = (\text{des}, \text{inv}) \pi$ .

*Proof.* The proof is easily constructed by induction on  $k$ , following along the lines of the proof of Theorem 10 in [1]. In fact, our extension of that proof is so routine that we leave it as an exercise for the reader.  $\square$

*Remark 3.2.* We also note that a restriction of the map  $\psi$  yields Krattenthaler's bijection [7] between 321-avoiding and 312-avoiding permutations on  $\mathfrak{S}_n$  using Dyck paths (modified up to the suitable reversal and complementation symmetries).

## REFERENCES

- [1] P. Brändén, A. Claesson, Mesh patterns and the expansion of permutation statistics as sums of permutation patterns, *Electron. J. Combin.* **18(2)** (2011-2012), #P5.
- [2] D. Foata, Distributions eulériennes et mahoniennes sur le groupe des permutations, in *Higher Combinatorics*, M. Aigner, ed., vol. 19, D. Reidel Publishing Co., Dordrecht-Holland, 1977, pp. 27-49.
- [3] D. Foata, G. Han, Fix-Mahonian Calculus, III: a quadruple distribution, *Monatshefte für Mathematik* **154** (2008), 177-197.
- [4] I. Gessel, A coloring problem, *Amer. Math. Monthly* **98** (1991), 530-533.
- [5] I. Gessel, C. Reutenauer, Counting permutations with given cycle structure and descent set, *J. Combin. Theory Ser. A* **64** (1993), 189-215.
- [6] D. Kim, J. Zeng, A new decomposition of derangements, *J. Combin. Theory Ser. A* **96** (2001), 192-198.
- [7] C. Krattenthaler, Permutations with restricted patterns and Dyck paths, *Adv. Appl. Math.* **27** (2001), 510-530.
- [8] Z. Lin, On some generalized  $q$ -Eulerian polynomials, *Electron. J. Combin.* **20(1)** (2013), #P55.
- [9] P.A. MacMahon, *Combinatory Analysis*, 2 volumes, Cambridge University Press, London, 1915-1916. Reprinted by Chelsea, New York, 1960.
- [10] J. Shareshian, M. Wachs,  $q$ -Eulerian polynomials: excedance number and major index, *Electron. Res. Announc. Amer. Math. Soc.* **13** (2007), 33-45.
- [11] M. Wachs, Personal communication.



DEPARTMENT OF MATHEMATICS, HOWARD UNIVERSITY, WASHINGTON, DC 20059 USA  
*E-mail address:* [aburstein@howard.edu](mailto:aburstein@howard.edu)  
*URL:* <http://www.alexanderburstein.org>